An algebraic approach to laying a ghost to rest

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Abstract: In the recent literature there has been a resurgence of interest in the fourth-order field-theoretic model of Pais-Uhlenbeck [15], which has not had a good reception over the last half century due to the existence of *ghosts* in the properties of the quantum mechanical solution. Bender and Mannheim [3] were successful in persuading the corresponding quantum operator to 'give up the ghost'. Their success had the advantage of making the model of Pais-Uhlenbeck acceptable to the physical community and in the process added further credit to the cause of advancement of the use of $\mathcal{PT}$ symmetry. We present a case for the acceptance of the Pais-Uhlenbeck model in the context of Dirac's theory by providing an Hamiltonian which is not quantum mechanically haunted. The essential point is the manner in which a fourth-order equation is rendered into a system of second-order equations. We show by means of the method of reduction of order [12] that it is possible to construct an Hamiltonian which gives rise to a satisfactory quantal description without having to abandon Dirac.
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1 Introduction

In one of the great books of Science Dirac [5] provided a theoretical environment for the rapidly developing subject of Quantum Mechanics based upon an interpretation of Mechanics elaborated by Hamilton [7] almost a century earlier in terms of operators. In the process and following the audacious line of thinking established by his predecessor at Bristol, Oliver Heaviside, Dirac introduced the distributions which were the bane of mathematicians until Laurent Schwartz [16, 17] provided a theoretical justification and George Temple [18] clarified that justification. In his monograph Dirac took the classical Hamiltonian as the energy and it is by no means obvious that he ever considered the possibility of any other Hamiltonian function as the basis for his operators. Moreover one should emphasise that the energy was conserved so that the approach of Bateman [2] to the quantisation of the damped linear oscillator could scarcely be regarded as being within the purview of Dirac's theory[2].

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[2]Bateman's contribution seems already to have become lost by the following decade since the names often associated with this problem are Caldriola [4] and Kanai [8]. The question

There arose models of quantum mechanical systems which did not coincide with Dirac's *obiter dictum* of a Hamiltonian as the energy. Indeed the direct association of a Hamiltonian with such a model was impossible. A well-known example is the model of Pais-Uhlenbeck [15] in which the Action is given by

$$A = \tfrac{1}{2}\gamma \int \left\{ \ddot{z}^2 - \left(\Omega_1^2 + \Omega_2^2\right)\dot{z}^2 + \Omega_1^2\Omega_2^2 z^2 \right\} \mathrm{d}t \qquad (1.1)$$

so the Euler-Lagrange equation is of the fourth order, being

$$\ddddot{z} + \left(\Omega_1^2 + \Omega_2^2\right)\ddot{z} + \Omega_1^2\Omega_2^2 z = 0. \qquad (1.2)$$

To bring this model within the context of Hamiltonian Theory Mannheim and Davidson [10] and Mannheim [11] introduced a new variable, y, to describe the model in terms of a system with two degrees of freedom. The resulting Hamiltonian is [3]

$$H = \frac{p_y^2}{2\gamma} + p_z y + \frac{\gamma}{2}\left(\Omega_1^2 + \Omega_2^2\right)y^2 - \frac{\gamma}{2}\Omega_1^2\Omega_2^2 z^2. \qquad (1.3)$$

The problem with the model of Pais-Uhlenbeck is that it possesses 'ghost' states, *ie* the norm of the (quantum) state is negative. This is not acceptable mathematically as well as physically since the whole concept of a norm is rooted in the essence of being nonnegative. Consequently the model has been regarded as unphysical. Even in the representation as a two-degree-of-freedom system, (1.3), there are problems. According to Bender and Mannheim [3] there are two possibilities depending upon the operators which annihilate the groundstate. The first gives a negative norm[3]. The second avoids that problem by giving a spectrum which is unbounded from below. Neither option could be described as acceptable!

In the last decade or so there has been a considerable expansion of what has been termed $\mathcal{PT}$ quantum mechanics. This is not the place to enter into a disquisition on the subject. We refer the reader to the references given in Bender and Mannheim [3]. It suffices to observe that in $\mathcal{PT}$ quantum mechanics the concept of an hermitian operator in the sense of Dirac[4] is replaced by definitions based more in Physics. The $\mathcal{P}$ operator is that of reflection in space and $\mathcal{T}$ is the operation of time reversal.

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of how to deal properly with dissipation in Quantum Mechanics became a lively issue in the following decades and is an interesting study in itself, but it is not immediately germane to our present topic.

[3]We maintain the definition of norm in the sense of Dirac.

[4]This is the perfectly normal one in Mathematics of being invariant under the two processes of transposition of the matrix and the taking of the complex conjugate of the elements.

The case made by Bender and Mannheim [3] is that, if one interprets the Pais-Uhlenbeck model in terms of $\mathcal{PT}$ quantum mechanics, the problem of the existence of ghost states is removed. Thereby the validity of the model is enhanced.

In this Letter we do not argue against the use of $\mathcal{PT}$ quantum mechanics for it does seem to be able to explain many phenomena which experience difficulties when discussed in terms of the model of Dirac. However, we present a case for the acceptance of the Pais-Uhlenbeck model in the context of Dirac's theory. The essential point is the manner in which the fourth-order equation, (1.2), is rendered into a system of second-order equations. We show by means of the method of reduction of order [12] that it is possible to construct an Hamiltonian which gives rise to a satisfactory quantal description without having to abandon Dirac.

The plan of this Letter is simple. In the next Section we demonstrate the construction of a suitable Lagrangian by means of the technique of the method of reduction of order. The Hamiltonian follows in the natural way. In the following Section we briefly perform the obvious quantisation procedure in the manner of Dirac to demonstrate explicitly that we have a physically consistent description in the case of the Pais-Uhlenbeck model.

Before we commence it is appropriate that we remind the reader of the importance of symmetry in the analysis of differential equations, be they from classical or quantal origins. When one is dealing with Lagrangian, hence Hamiltonian, systems the importance of symmetry seems to be enhanced. One can make many Lagrangians for a given system [14]. They need not be identical in their properties in terms their symmetries. It is a present matter for investigation of the implications of a differing number of Noether symmetries for Lagrangians describing the same physical system.

2   Lagrangian description for the Pais-Uhlenbeck model

The method of reduction of order has been used in a number of papers in recent years (see [13] and references within) and we briefly summarise the method. Further details may be found in the papers cited. Given a system of ordinary differential equations of greater than the first order (at least one of the system), the system is replaced by another system of equations of the first order by the introduction of the requisite number of new dependent variables. At least one of the variables is removed so that there is at least one equation of the second order. This enables one to make sensible use of Lie's theory of continuous groups since the number of point symmetries is then finite. Armed with the symmetries one can then make a further analysis by means of the standard theory. In the present case we are concerned with the transition from a classical description to a

quantum mechanical description. Obviously symmetry plays an important role in both descriptions. Our initial problem in terms of the method of reduction of order of the fourth-order equation, (1.2), is to find a pair of second-order equations for which an obtainable Lagrangian description exists. Then we have the question of the determination of an Hamiltonian and is quantisation to an operator which has sensible properties.

We observe that generally speaking there are many possible ways in which a system of first-order equations can be constructed from the original fourth-order equation and then retreaded as a pair of second-order equations. The way[5] chosen by Mannheim and Davidson [10] and Mannheim [11] was demonstrated by Bender and Mannheim [3] to be unsatisfactory in terms of the prescriptions of Dirac[6]. Our systematic approach through the method of reduction of order demonstrates a more than somewhat different result.

We introduce new variables to render (1.2) as a system of four first-order equations. To maintain a consistent notation we write $z = w_1$ and continue as

$$\dot{w}_1 = w_2 \tag{2.1}$$
$$\dot{w}_2 = w_3 \tag{2.2}$$
$$\dot{w}_3 = w_4 \tag{2.3}$$
$$\dot{w}_4 = -\left(\Omega_1^2 + \Omega_2^2\right) w_3 - \Omega_1^2\Omega_2^2 w_1. \tag{2.4}$$

In the usual application of the method of reduction of order an ignorable coordinate can the eliminated, but in this instance we do not wish to remove t as the independent variable. As our intermediate aim is to construct an Hamiltonian, we wish to rewrite the system of four first-order equations as a pair of second-order equations as a prelude to the construction of a first-order Lagrangian in two dependent variables. If we eliminate w_2 and w_4, the system, (2.1-2.4), becomes

$$\ddot{w}_1 = w_3 \tag{2.5}$$
$$\ddot{w}_3 = -\left(\Omega_1^2 + \Omega_2^2\right) w_3 - \Omega_1^2\Omega_2^2 w_1. \tag{2.6}$$

Continuing in the spirit of the method of reduction of order we introduce two more suitable dependent variables by means of

$$w_1 = r_1 - r_2 \tag{2.7}$$
$$w_3 = -\Omega_1^2 r_1 + \Omega_2^2 r_2, \tag{2.8}$$

from which it is evident that we are dealing with the case $\Omega_1 > \Omega_2$. Equations (2.7) and (2.8) are written in terms of the new variables and rearranged to give

$$\ddot{r}_1 = -\Omega_1^2 r_1 \tag{2.9}$$

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[5]Evidently without an appreciation of the general principles of the method of reduction of order.

[6]Although it is not stated so in [3], in [10] and [11] we are informed that the Hamiltonian description is obtained using the method of Ostrogradsky, as described in the classical text by Whittaker [19].

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$$\ddot{r}_2 = -\Omega_2^2 r_2, \tag{2.10}$$

which obviously describes a two-dimensional anisotropic oscillator.

A first-order Lagrangian for system (2.9,2.10) is

$$L = \tfrac{1}{2}\left(\dot{r}_1^2 + \dot{r}_2^2 - \Omega_1^2 r_1^2 - \Omega_2^2 r_2^2\right) \tag{2.11}$$

from which it is obvious that the canonical momenta are

$$p_1 = \dot{r}_1 \quad \text{and} \quad p_2 = \dot{r}_2 \tag{2.12}$$

and so the Hamiltonian is

$$H = \tfrac{1}{2}\left(p_1^2 + p_2^2 + \Omega_1^2 r_1^2 + \Omega_2^2 r_2^2\right). \tag{2.13}$$

We note that the Hamiltonian of (2.13) belongs to the class of classical Hamiltonians envisaged by Dirac.

3 Quantum Mechanical Formulation

Since the Hamiltonian, (2.13), is under the ægis of Dirac's canon, we may apply the standard methods of quantisation to obtain the Schrödinger equation

$$2i\frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial r_1^2} - \frac{\partial^2 u}{\partial r_2^2} + \left(\Omega_1^2 r_1^2 + \Omega_2^2 r^2\right)u. \tag{3.1}$$

This has seven Lie point symmetries[7]. They are

$$\Gamma_{\pm 1} = \exp[\pm i\Omega_1 t]\left\{\mp\partial_{r_1} + \Omega_1 r_1 u\partial_u\right\} \tag{3.2}$$

$$\Gamma_{\pm 2} = \exp[\pm i\Omega_2 t]\left\{\mp\partial_{r_2} + \Omega_2 r_2 u\partial_u\right\} \tag{3.3}$$

$$\Gamma_3 = i\partial_t \tag{3.4}$$

$$\Gamma_4 = u\partial_u \tag{3.5}$$

$$\Gamma_5 = f(t, r_1, r_2)\partial_u, \tag{3.6}$$

where f is a solution of (3.1)[8].

Since the Hamiltonian, (2.13), is separable in the variables which we have selected, the determination of Dirac's creation and annihilation operators follows

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[7]Five of these correspond to the Noether point symmetries of the Lagrangian, (2.11), as one would expect [9]. The remaining two are generic to linear evolution equations. The corresponding system, (2.9,2.10), possesses seven Lie point symmetries [6]. Five of these are the same as the Noether point symmetries. The additional two are a consequence of the linearity of the equations in the dependent variables.

[8]Naturally equation (3.1) comes with boundary conditions. The symmetry Γ_5 is not confined to being in terms of functions which satisfied the boundary conditions.

immediately from the symmetries (3.2) and (3.3) which are their progenitors [9, 1]. The energy follows as the eigenvalue of the action of Γ_3 on a solution. The groundstate, as an explicitly time-dependent function, is obtained using Γ_1 and Γ_2 as follows.

The invariants of Γ_1 are determined by the solution of the associated Lagrange's system

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}r_1}{-1} = \frac{\mathrm{d}r_2}{0} = \frac{\mathrm{d}u}{\Omega_1 r_1 u} \tag{3.7}$$

and are t, r_2 and $v = u \exp\left[\frac{1}{2}\Omega_1 r_1^2\right]$. In terms of these invariants Γ_2 is $\exp[i\Omega_2 t]\left(-\partial_{r_2} + \Omega_2 r_2 v \partial_v\right)$. The solution of the corresponding associated Lagrange's system gives the invariants t and $w = v \exp\left[\frac{1}{2}\Omega_2 r_2^2\right]$. The double reduction of (3.1) to an ordinary differential equation is achieved by writing

$$u = \exp\left[-\frac{1}{2}\left(\Omega_1 r_1 + \Omega_2 r_2^2\right)\right] f(t). \tag{3.8}$$

The reduced equation is

$$\frac{\dot{f}}{f} = \frac{1}{2i}\left(\Omega_1 + \Omega_2\right), \tag{3.9}$$

where the overdot denotes differentiation with respect to t, and has the solution $f(t) = \exp\left[-\frac{1}{2}\left(\Omega_1 + \Omega_2\right) it\right]$.

The groundstate wavefunction is, up to the normalisation factor,

$$u_0 = \exp\left[-\frac{1}{2}\left(\Omega_1 + \Omega_2\right) it - \frac{1}{2}\left(\Omega_1 r_1 + \Omega_2 r_2^2\right)\right]. \tag{3.10}$$

The action of Γ_3 gives the groundstate energy through

$$\Gamma_3 u_0 = \frac{1}{2}\left(\Omega_1 + \Omega_2\right) u_0 \tag{3.11}$$

$$\implies \quad E_0 = \frac{1}{2}\left(\Omega_1 + \Omega_2\right). \tag{3.12}$$

Further eigenstates are obtained by the actions of Γ_{-1} and Γ_{-2} which act as creation operators for the time-dependent Schrödinger equation, (3.1). Since these are simply combinations of the eigenstates of two harmonic oscillators, there are no difficulties with the positivity of the energy spectrum, which is obtained by the action of Γ_3 on the created eigenstates, and the nonnegativity of the norm of the wavefunction in the sense of Dirac.

4   Observations

It has not been our intention to belittle the value of the use of $\mathcal{PT}$ symmetry in the resolution of some questionable problems in quantum mechanics. What we have specifically desired to demonstrate is that the model of Pais-Uhlenbeck can

be rendered into an Hamiltonian form which can be quantised and lead to results after a mathematical analysis which are consistent with the physical principles underlying the model. In this respect we are guided by the principle of the maintenance of symmetry in going from the classical model to the corresponding Schrödinger equation. The second-order Lagrangian for the Action Integral in (1.1) possesses five Noether point symmetries. After the application of the method of reduction of order to obtain the Lagrangian (2.11) we find that this Lagrangian possesses seven Noether point symmetries. This increase is not to be unexpected since we have essentially introduced generalised symmetries[9]. In our approach these Noether symmetries are preserved in the transition to quantum mechanics as the nongeneric symmetries of the Schrödinger equation corresponding to *our* Hamiltonian. It is not at all obvious that this be the case for the Hamiltonian used in [10], [11] and [3].

We would propose as a general principle that any technique of quantisation from a corresponding classical system be such that there is preservation of the Noether point symmetries of the classical Lagrangian in the nongeneric Lie point symmetries of the corresponding Schrödinger equation. It is unfortunate that at this stage we cannot remove the 'would' from that which we propose since it is a very general question, but we are working on it.

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[9]The application of the method of Ostrogradsky does the same in that the second dependent variable introduced is $\dot{z}$. This does not mean that the construction of the two first-order Lagrangians are equivalent with respect to point symmetries.

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